NYU-TH-05/06/01

Classically Constrained Gauge Fields and Gravity

Gregory Gabadadze and Yanwen Shang

Center for Cosmology and Particle Physics
Department of Physics, New York University, New York, NY, 10003, USA

Abstract

We study gauge and gravitational field theories in which the gauge fixing conditions are imposed as constraints on classical fields. Quantization of fluctuations can be performed in a BRST invariant manner, while the main novelty is that the classical equations of motion admit solutions that are not present in the standard approach. Although the new solutions exist for both gauge and gravitational fields, one interesting example we consider in detail is constrained gravity endowed with a nonzero cosmological constant. This theory, unlike General Relativity, admits two maximally symmetric solutions one of which is a flat space, and another one is a curved-space solution of GR. We argue that, due to BRST symmetry, the classical solutions obtained in these theories are not ruined by quantum effects. We also comment on massive deformations of the constrained models. For both gauge and gravity fields we point out that the propagators of the massive quanta have soft ultraviolet behavior and smooth transition to the massless limit. However, nonlinear stability may require further modifications of the massive theories.

1 Introduction and summary

To quantize an electromagnetic field one could fix a gauge by imposing an operator constraint on physical states of the theory. For instance, in the Gupta-Bleuler approach one postulates

$$G(A)|\Psi\rangle = 0\,, \tag{1}$$

where $G(A)$ denotes a function of the gauge field operator A or its derivative, and $|\Psi\rangle$ is an arbitrary *physical* state of the theory.

Alternatively, one could choose to impose the constraint on a gauge field already in a classical theory in such a way that (1) is automatic upon quantization [1, 2]. This can be achieved, e.g., by introducing into the classical action of the theory a Lagrange multiplier λ times the function $G(A)$

$$\int d^4x\,\lambda(x)\,G(A(x))\,. \tag{2}$$

The field λ has no kinetic or potential terms. Variation of the action w.r.t. λ gives a constraint $G(A(x)) = 0$, which is just a classical counterpart of (1). Small fluctuations of the fields in this theory can consistently be quantized for Abelian [1, 2, 3] as well as non-Abelian and gravitational fields [4]. The resulting theories can be completed to be invariant under the Becchi-Rouet-Stora-Tyutin (BRTS) transformations [5], and a Hilbert space of physical states can be defined by requiring that the states carry zero BRST and ghost charges [4]. Quantum effects in the resulting theory are identical to those of the Gupta-Bleuler approach.

Nevertheless, there is one difference in using the classically constrained theory that has not been explored. This difference could be seen in classical equations of motion. Variation of the action w.r.t. the gauge fields gives an equation of motion in which there are new terms proportional to λ and/or its derivatives. Hence, classical field equations are modified, and, depending on allowed boundary conditions, new solutions could emerge.

This becomes especially important for gravity. General Relativity (GR) with a non-zero cosmological constant does not admit Minkowski space as a solution of equations of motion. We will discuss classically constrained General Relativity (CGR) in section 3 and show that the latter does admit flat space as a solution even if the cosmological constant is not zero. This difference is clearly very important.

Unimodular gravity (UGR) [6] is an interesting example of a *partially constrained* theory. In UGR the full reparametrization invariance of GR is restricted to a subgroup of volume preserving transformations. One practical difference between UGR and GR is that the cosmological constant problem in UGR is somewhat relaxed. This is because UGR with a cosmological constant admits an infinite number of maximally symmetric solutions labeled by the value of the space-time curvature. However, this does not explain why one should choose the desirable (almost) flat solution among a continuum of maximally symmetric ones. Another issue in UGR

is related to quantum loops of matter and gravity. It is likely that in this theory the Lagrange multiplier λ acquires quadratic terms via the quantum loops (see, section 3); if so, new interactions would be needed to maintain the classical solutions of UGR in a quantum theory.

Constrained GR improves on the above-mentioned aspects: (i) it admits only *two* maximally symmetric solutions – one with zero curvature, and another one with the curvature obtained in GR; (ii) its classical properties, due to the BRST invariance of that model, are not modified by the quantum loop effects.

Reparametrization invariance in CGR is completely constrained. We will show in section 3, that one can still define a *locally inertial* reference frame in a small region around an arbitrary space-time point. This is because the constraint that fixes gauge in the whole space, allows, in the neighborhood of a given point, for the point-dependent gauge transformations that locally eliminate effects of gravity. Thus the equivalence principle is preserved.

Do the constrained theories solve the "old" cosmological constant problem [7]? In UGR the answer is negative because the theory admits an infinite number of maximally symmetric solutions. Clearly, in CGR, there still exists a conventional de Sitter solution of GR which can be used for inflation in the early universe. But the question is whether there exists an infinite number of other non-maximally symmetric solutions in the theory with a cosmological constant. If some of these solutions are physical, one should understand why in our Universe the (almost) flat solution is preferred. If, on the other hand, the non-maximally symmetric solutions can be disregarded for one reason or other, then CGR could be a good starting point for trying to accommodate inflation in the early universe and still solve the "old" cosmological constant problem. These important issues, including the question of stability of the new solutions, are being studied in [8]. The purpose of the present work is to investigate CGR to determine whether it is a consistent low-energy quantum field theory, which by itself is a legitimate theoretical question. Concrete applications of this model will be discussed in [8].

Even in the most optimistic scenario for CGR, one should explain why the space-time curvature is not exactly zero but $\sim H_0^2 \sim (10^{-42}\ \mathrm{GeV})^2$, as suggested by recent observations. Where could this scale come from? One possibility is to introduce a pseudo Nambu-Goldstone boson potential [9] that gives rise to required "dark energy" [1]. Another way is to try introducing a graviton mass $m_g \sim H_0$. Although both of the above approaches postulate the existence of a new small scale, this scale is stable w.r.t. quantum corrections (i.e., it is technically natural). In this regard, we briefly discuss in the present work massive deformations of classically constrained gauge and gravitational theories. We will find that the UV behavior of the propagators of massive gauge and gravitational quanta are softened. Although these results are encouraging, at this stage we still lack an understanding of whether the non-linear unitarity of the S-matrix on a Hilbert space of physical states can be pre-

[1]In this case though, one needs a VEV of a scalar to be somewhat higher than the Planck mass.

served using the BRST and ghost charges of these models, or whether some further modifications may be needed. Studies on these issues will be reported elsewhere.

The work is organized as follows: in section 2 we discuss a constrained theory of a photon and its classical equations and solutions. In section 3 we discuss a classically constrained theory of gravity (CGR). We find that CGR with a cosmological constant has a remarkable property – it admits two maximally symmetric solutions one of which is a flat space. We find general cosmological solutions in CGR as well as the expression for the Schwarzschild metric. An important question that is also addressed in section 3 is that of radiative stability. Using the BRST invariant version of the theory we argue that quantum corrections do not ruin the obtained classical results. Massive deformations of CGR are also briefly discussed in section 3. In Appendix A we discuss the spectrum of the constrained Abelian gauge theory in various approaches, including Stückelberg's method. We also look at the massive deformation of this model pointing out that the massive propagator, unlike in the Proca theory, has smooth UV behavior and a nonsingular massless limit. Appendix B deals with constrained non-Abelian gauge fields. After briefly discussing classical equations, we study the spectrum of the theory. Comments on massive deformations of the non-Abelian theories are also included. The results are similar to those of the Abelian case. Some parts of Appendix A and B are of a review character, but we felt that including these discussions would make our presentation more complete.

2 Constrained photon

As an instructive example we consider electrodynamics with an imposed classical constraint. We call this model constrained QED (CQED) even though we will only quantize it later. We start with the Lagrangian density

$$l = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,+\, A_\mu J^\mu \,+\, \lambda(\partial_\mu A^\mu)\,. \tag{3}$$

Here $\mu, \nu = 0, 1, 2, 3$; J^μ is a current, and λ is a Lagrange multiplier (our choice of the Lorentzian signature is "mostly negative").

We start by studying classical properties of this theory. The equations of motion that follow from the above Lagrangian are

$$\partial^\nu F_{\nu\mu} + J_\mu - \partial_\mu\lambda = 0\,, \tag{4}$$

$$\partial_\mu A^\mu \;=\; 0\,. \tag{5}$$

Taking a derivative of Eq. (4) one obtains the following relation

$$\Box\lambda = \partial^\mu J_\mu\,. \tag{6}$$

If the current J_μ is conserved, λ is a harmonic function, $\Box\lambda = 0$. One particular solution of this equation is $\partial_\mu\lambda = C_\mu$, where C_μ denotes an arbitrary space-time

constant four-vector. Physically, the effective conserved current to which the gauge field is coupled in (3) is

$$J_\mu^{\rm eff} = J_\mu - \partial_\mu \lambda = J_\mu - C_\mu \,. \tag{7}$$

The last term on the r.h.s. of (7) acts as a constant background current density determined by a vector C_μ. Since C_μ is an integration constant, there is a continuous set of C_μ's that one could choose from. Setting the value of C_μ is equivalent of choosing the corresponding boundary conditions. The equations of motion (4) and (5) were derived by varying the action corresponding to (3) with the following boundary conditions

$$\delta A_\mu|_{\rm boundary} = 0, \quad \delta\lambda|_{\rm boundary} = \text{finite function}\,. \tag{8}$$

Therefore, all the solutions should obey (8). To demonstrate that such solutions exist let us consider a simple example of a spherically symmetric localized charge density for which $J_0 = \rho\,\theta(r_0 - r)$, $J_i = 0$. Here ρ is a constant charge density, $\theta(r)$ denotes the step function, r is the radial coordinate, and r_0 is the radius of the charge distribution. We substitute this source into the RHS of (4). In addition we chose a solution for the Lagrange multiplier to be $\partial_0\lambda \equiv C_0 = \rho$, and $\partial_j\lambda \equiv C_j = 0$. For this source a solution of Eqs. (4) and (5), and the corresponding electric field read

$$\begin{aligned} A_0 &= -\frac{\rho r_0^3}{3r} - \frac{\rho r^2}{6} \quad \text{for} \;\; r \geq r_0\,, \quad A_0 = -\frac{\rho r_0^2}{2} \quad \text{for} \;\; r \leq r_0\,, \\ \vec{E} &= \frac{\rho}{3}\,(1 - \frac{r_0^3}{r^3})\,\theta(r - r_0)\vec{r}\,. \end{aligned} \tag{9}$$

The above solution satisfies the boundary conditions (8) at $r = r_0$. In the conventional electrodynamics the source $J_0 = \rho\theta(r_0 - r)$, $J_i = 0$, yields an electric field that is well-known and differs from (9). The origin of this difference is clear – in CQED the quantity J_μ that is specified in the action *is not the whole source* producing the gauge field! An additional integration constant appears in the equations of motion and the total source is $J_\mu^{\rm eff}$ (7). The above theory reduces to conventional electrodynamics when we choose $\lambda = 0$. This corresponds to what we measure in ordinary experiments.

Similar solutions with a nonzero value of the Lagrange multiplier will play an important role for gravity with a nonzero cosmological constant. These will be discussed in section 3 (non-Abelian gauge fields are discussed in Appendix B).

So far we have not emphasized the fact that the Lagrangian (3) is not gauge invariant. In fact, variation of (3) under the gauge transformation $\delta A_\mu = \partial_\mu\alpha(x)$ vanishes (up to a surface term) for configurations satisfying $\Box\lambda = 0$, therefore (3) has gauge invariance when the λ field is *on-shell*, even though the gauge field could be off-shell. Given that the λ is not propagating, and remains such at the quantum level (see below), this suggests two physical degrees of freedom for a photon, while

off-shell there are four degrees of freedom. This will be established more rigorously below. As it is shown in Appendix A, the two extra off-shell degree of freedom are decoupled from conserved sources and have no relevance in the Abelian case.

The quantum theory of CQED could be approached in a number of different ways. One could restore first a manifest gauge invariance of (3) using the Stückelberg method, and then quantize the resulting gauge invariant theory. We will discuss this in Appendix A and B for Abelian and non-Abelian gauge fields respectively.

In the Feynman integral formulation one could think of the constrained approach as follows: the gauge and auxiliary fields can be decomposed in their classical and quantum parts, $A = A_{cl} + \delta A$, and $\lambda = \lambda_{cl} + \delta\lambda$. For classical solutions, A_{cl} and λ_{cl}, we allow boundary conditions that are different from the conventional ones. In particular, we allow for a nonzero solution of the $\Box\lambda = 0$ equation, a zero-mode. This is an unconventional step, since the usual FP term that appears in the path integral is the determinant of the operator acting on λ, that is $det(\Box)$ in this particular case. Because of the zero-mode, the determinant, $det(\Box)$, would have been zero, leading to an ill-defined partition function. However, a right way to formulate the path integral is to separate the zero-mode, and integrate only w.r.t. the small fluctuations for which $det(\Box)$ is non-zero, and for which the conventional radiation boundary conditions are imposed.

This is the procedure that we will be assuming throughout the text. Then, quantization of the fluctuations δA and $\delta\lambda$ over the classical background A_{cl}, λ_{cl} can be performed in a BRST invariant way. That is, one could start by postulating BRST invariance of the Lagrangian for the fluctuations by adding in the Faddeev-Popov (FP) ghosts without any reference to a local gauge symmetry, but elevating BRST invariance to a fundamental guiding principle in constructing the Lagrangian. The resulting BRST symmetric theory of small fluctuations could be quantized in a conventional way. This is what we summarize below. For simplicity of presentation we will replace δA_μ and $\delta\lambda$ by A_μ and λ keeping in mind that these are fluctuations over a classical background (the same replacement will be assumed for non-Abelian and gravity fields considered in the next sections and Appendices).

In a conventional Feynman integral approach the measure in the path integral should be modded out by the gauge equivalent classes. The FP trick does this job by introducing the gauge-fixing term along with the FP ghosts. In this regard the following natural question arises – since (3) (or its non-Abelian counterpart) is not gauge invariant why do we need to introduce the FP ghosts in the theory? Naively, it would seem that we should use the path integral

$$\int DA\, \delta(G(A)) \exp\left(i \int l_{\text{Gauge fields}} \right) , \tag{10}$$

instead of the one with the FP ghosts

$$\int DA\, \delta(G(A)) \det \left| \frac{\delta G^\omega}{\delta\omega} \right| \exp\left(i \int l_{\text{Gauge fields}} \right) , \tag{11}$$

where G is a gauge-fixing condition (for instance it could be that $G = \partial_\mu A^\mu$), and G^ω is gauge transformed G with ω being the transformation parameter (the above definition is consistent as far as we do not include the zero modes in the integration and $det(\Box)$ is nonzero, as we discussed above). In QED the difference is irrelevant because the FP ghost are decoupled from the rest of the physics, however, in the case of non-Abelian fields (10) would lead to a non-unitary theory. Because of the absence of gauge invariance in (3), the argument to introduce the FP ghosts in the path integral of (3) cannot be the same as in the conventional case. Nevertheless, the FP ghosts can be motivated by a symmetry argument, namely by requiring the BRST invariance of the constrained action.

Let us modify (3) by adding the FP ghost c and anti-ghost $\bar{c}$ which are Grassmann variables, $c^2 = (\bar{c})^2 = 0$ and $\bar{c}^+ = \bar{c},\ c^+ = c$. The Lagrangian reads as follows

$$l = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + \lambda(\partial_\mu A^\mu) - i\bar{c}\Box c. \tag{12}$$

Since on the classical backgrounds considered here the FP ghost fields vanish, the classical properties of (3) and (12) are identical[2]. The Lagrangian (12), however, is invariant under the following continuous BRST transformations:

$$\delta A_\mu = i\zeta\partial_\mu c\,, \quad \delta c = 0, \quad \delta\bar{c} = \lambda\zeta\,, \quad \delta\lambda = 0\,, \tag{13}$$

where ζ is a coordinate independent Grassmann transformation parameter such that $(\zeta c)^+ = c^+\zeta^+ = c\zeta$. The Lagrangian (12) gives the path integral that is identical to that of the conventional approach (11) which takes the form

$$\int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}\bar{c}\mathcal{D}c \exp\left[i\int -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + A^\mu J_\mu + \lambda(\partial_\mu A^\mu) - i\bar{c}\Box c\right]. \tag{14}$$

Notice that the rationale for writing down (14) and the Lagrangian (12) is different from the motivation that led to the path integral (11). In the constrained approach the rules for constructing a Lagrangian and path integral are:

(1) Impose classical constraints on fields using the Lagrange multiplier technique. Find the corresponding zero-modes and treat them separately from small fluctuations of the fields.

(2) Introduce the FP ghosts to obtain the BRST invariance of the gauge non-invariant theory.

(3) Use this Lagrangian to set up the path integral in a straightforward way.

The presence of the BRST symmetry guarantees that all the Ward-Takahashi identities (the Slavnov-Taylor identities in the non-Abelian case) of the conventional theory are preserved in the constrained approach, even though the classical equations of motion in this approach are different as discussed above.

[2] In general, a ghost-antighost bilinear could have nonzero expectation values on certain states, however, these stats do not satisfy the zero BRST and Ghost charge conditions, see below.

Let us now turn to the loop corrections that emerge in (12). In particular we would like to make sure that no kinetic or potential terms are generated for λ. We do have a symmetry $\lambda(x) \to \lambda(x) + \beta(x)$, where β is an arbitrary function, w.r.t. which (12) is invariant. Kinetic or potential terms of λ would break it. The question is whether this symmetry is preserved by the loop corrections. To address this issue we calculate the propagator of the gauge fields. This can be done in a few different ways. The easiest one is to add a fictitious term $-\frac{1}{2}\gamma\lambda^2$ to the Lagrangian density and then take the limit $\gamma \to 0$

$$\begin{aligned} Z[J_\mu] &\propto \lim_{\gamma\to 0} \int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}\bar{c}\mathcal{D}c \exp\left[i\int l_g + A^\mu J_\mu + \lambda(\partial_\mu A^\mu) - \frac{1}{2}\gamma\lambda^2 - i\bar{c}\Box c\right] \\ &\propto \lim_{\gamma\to 0} \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \exp\left[i\int l_g + A^\mu J_\mu + \frac{1}{2\gamma}(\partial_\mu A^\mu)^2 - i\bar{c}\Box c\right], \end{aligned} \tag{15}$$

where $l_g \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The propagator is obtained following the standard procedure and the result is

$$\Delta^{\mu\nu} = -\lim_{\gamma\to 0} \frac{\eta^{\mu\nu} - (1+\gamma)\partial^\mu\partial^\nu/\Box}{\Box} = -\frac{\eta^{\mu\nu} - \partial^\mu\partial^\nu/\Box}{\Box}, \tag{16}$$

which coincides with the standard transverse QED propagator in Landau gauge[3]. It is straightforward to check that loop corrections preserve the transversality of the gauge field propagator. All the diagrams that renormalize the propagator consist of the standard bubble diagrams produced by the contractions between two currents. Because the current to which the gauge field is coupled is conserved, the propagator remains transverse to all loops

$$\partial^\mu\lambda\,\langle A_\mu\, A_\nu\rangle\partial^\nu\lambda \sim \partial^\mu\lambda(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial^\nu\lambda = 0\,. \tag{17}$$

As a result, loops cannot generate the kinetic term for λ.

Last but not least, the BRST symmetry of (12) can be used to define the Hilbert space of physical states by imposing the standard zero BRST- and Ghost- charge conditions $Q^{\rm BRST}|{\rm Phys}>= 0$, $Q^{\rm Ghost}|{\rm Phys}>= 0$. In the Abelian case this reduces [4] to the Nakanishi-Lautrup condition $\lambda^{(-)}|{\rm Phys}>= 0$ [1, 2] (here $\lambda^{(-)}$ denotes a negative frequency part of a fluctuation of λ over a classical background) which ensures that the quantum of λ is not in a set of physical *in* and *out* states of the theory.

[3]Here and below we always assume the Feynman (causal) prescription for the poles in the propagators.

3 Constrained models of gravity

There exists a modification of GR, unimodular gravity (UGR) [6], that *partially* restricts its gauge freedom and yet reproduces the observable results of the Einstein theory. In UGR reparametrization invariance is restricted to the volume-preserving diffeomorphisms that keep the value of $\sqrt{g}$ intact ($g \equiv |det g_{\mu\nu}|$). Such a theory can be formulated by using the Lagrange multiplier λ (we set $8\pi G_N = 1$, unless indicated otherwise):

$$\mathcal{L} = -\frac{\sqrt{g}}{2}R - \lambda(\sqrt{g} - 1) + \mathcal{L}_M, \tag{18}$$

where $\mathcal{L}_M$ is the Lagrangian of the matter fields. The equations of motion and Bianchi identities of this theory require λ to be an arbitrary space-time constant [6]. As a result, UGR is equivalent at the classical level to GR except that cosmological term becomes an arbitrary integration constant. Since the latter can take any value, there are an infinite number of solutions parametrized by a constant scalar curvature. Such a theory, at least at the classical level, seems to be more favorable than GR – for an arbitrary large value of the vacuum energy in the Lagrangian (arising, e.g., from particle physics) one is always able to find an almost-flat solution. Of course this still does not explain why one should choose the desirable almost-flat solution among a continuum of maximally symmetric ones. This is one aspect of UGR on which one would like to improve.

Another, perhaps more pertinent question in UGR emerges when one considers quantum loops of matter and gravity. Inspection of loop diagrams (see section 4.2) suggest that the Lagrange multiplier λ in (18) would acquire the mass (λ^2) as well as kinetic ($(\partial_\mu \lambda)^2$) terms due to the quantum effects. As a result, λ would cease to be an auxiliary field, and all the classical results of UGR would have to be reconsidered.

We will discuss a model that completely constrains reparametrization invariance of GR. This theory has the following two important properties: (I) It allows, like UGR does, an adjustment of the cosmological constant via the integration constant mechanism, but only admits two maximally symmetric solutions, one of which is a flat space. (II) Unlike UGR its classical properties are stable w.r.t. quantum corrections, i.e., the Lagrange multiplier remains an auxiliary field even in the quantum theory. This theory also preserves the equivalence principle.

3.1 Constrained gravity and the cosmological constant

Consider the following Lagrangian

$$\mathcal{L} = -\frac{\sqrt{g}}{2}R + \sqrt{g} g^{\mu\nu} \partial_\mu \lambda_\nu + \mathcal{L}_M + \text{surface terms}. \tag{19}$$

Here, λ_ν is a vector that serves as a Lagrange multiplier and $\mathcal{L}_M$ denotes the Lagrangian of other fields which can also include a vacuum energy term (the cosmological constant) produced by classical and/or quantum effects. Versions of this

model have been discussed in the literature previously (see, e.g., the last reference in [4]) with the purpose of introducing the de Donder gauge-fixing condition in the context of quantization of gravity. Here, instead, we regard this model as a classical theory, which is subsequently quantized, but the classical equations of which could admit rather interesting solutions that are absent in GR. Thus, we first concentrate on classical effects, leaving the discussion of quantum corrections for the next subsection[4].

Among others, we will be considering below solutions with fixed boundaries. For such solutions one should add to the Einstein-Hilbert action the Gibbons-Hawking boundary term. Moreover, the boundary conditions that we allow for are:

$$\delta g_{\mu\nu}|_{\text{boundary}} = 0, \quad \text{and} \quad \delta\lambda_\mu|_{\text{boundary}} = 0\,. \tag{20}$$

Under these conditions the variation w.r.t. the metric gives

$$G_{\mu\nu} - (\partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu) + g_{\mu\nu}\partial^\sigma\lambda_\sigma = T_{\mu\nu}\,. \tag{21}$$

In addition to this, we get a constraining equation by varying the action w.r.t. the Lagrange multiplier

$$\partial_\mu(\sqrt{g}\, g^{\mu\nu}) = 0\,. \tag{22}$$

The above model is similar in spirit to a constrained gauge theory discussed in the previous section. It is easily checked that (22) is equivalent to the condition

$$\sqrt{g}\Gamma^\alpha_{\mu\nu} g^{\mu\nu} = 0\,, \tag{23}$$

and, in the linearized approximation, gives rise to de Donder (harmonic) gauge fixing of linearized GR. Due to this condition, $\nabla_\mu A^\mu = g^{\mu\nu}\partial_\mu A_\nu \equiv \partial^\mu A_\mu$, where ∇_μ is a covariant derivative acting on an arbitrary four-vector A_μ.

The constraint (22) (or (23)) fixes completely reparametrization invariance of the theory. How is then the equivalence principle recovered? A relevant property of (22) (or (23)) is this: for any given point $x^\mu = x_0^\mu$ in the coordinate system $\{x^\mu\}$, it allows for the x_0^μ-dependent coordinate transformations that eliminate the connection in a small neighborhood of this point. These transformations can be written as

$$x'^\mu = x^\mu + \frac{1}{2}\Gamma^\mu_{\alpha\beta}(x_0)(x-x_0)^\alpha(x-x_0)^\beta\,. \tag{24}$$

It is straightforward to check that for (24) $g^{\alpha\beta}(x)|_{x=x_0} = g'^{\alpha\beta}(x')|_{x'=x_0}$, and

$$\Gamma^\mu_{\alpha\beta}(x)|_{x=x_0} = \Gamma'^\mu_{\alpha\beta}(x')|_{x'=x_0} + \Gamma^\mu_{\alpha\beta}(x)|_{x=x_0}\,. \tag{25}$$

[4]One could choose to impose a different constraint using the Lagrange multiplier in (19) (for instance, an axial-gauge constraint). As long as it is an acceptable gauge-fixing condition for small fluctuations, and the corresponding FP ghosts are taken care of consistently (in the axial-gauge the FP ghost are not needed) the quantum effects of the fluctuations won't depend on the choice of this constraint. However, the classical solutions could differ for different constraints. Here we choose the constraint that is Lorentz-invariant.

As a result, $\Gamma'^{\mu}_{\alpha\beta}(x')|_{x'=x_0} = 0$, and at this point the metric can simultaneously be brought to the Minkowski form. The transformation (24), on the surface of (22) (or (23)), is trivially consistent with (22) (or (23))

$$g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}(x)|_{x=x_0} = g'^{\alpha\beta}(x')\Gamma'^{\mu}_{\alpha\beta}(x')|_{x'=x_0} = 0. \tag{26}$$

Since x_0 was arbitrary, the above arguments can be repeated for any other point in space-time. One should emphasize again that the coordinate transformation (24) is *point-specific*, i.e., at different points of space-time one should perform transformations that depend parametrically on that very point. This is why they are not gauge transformations in the entire space-time. Summarizing, although (22) (or (23)) picks a global coordinate frame, it allows for the point-dependent coordinate transformations that can eliminate gravity locally. Hence, the equivalence principle. The local Lorentz transformations are also preserved.

To obtain the equation which the Lagrange multiplier has to satisfy we apply a covariant derivative to both sides of (21). This is subtle since the second term on the l.h.s. of (21) is not a tensor and we should define the action of a covariant derivative on this object. We adopt the following straightforward procedure: apply to both sides of (21) the operator

$$\nabla^{\alpha} \equiv g^{\alpha\mu}\left(\partial_\mu - \Gamma^{*}_{*\mu} - \Gamma^{*}_{*\mu}\right), \tag{27}$$

where the standard index arrangement should be used in place of the asterisks even if the two-index object on which this operator is acting does not transform as a tensor. Then, using the Bianchi identities and covariant conservation of the stress-tensor we obtain:

$$g^{\mu\alpha}\nabla_\mu(\partial_\alpha\lambda_\nu + \partial_\nu\lambda_\alpha) = \nabla_\nu\partial^\sigma\lambda_\sigma\,. \tag{28}$$

The above equation can be simplified substantially due to (22). The left hand side of (28) can be reduced to

$$g^{\mu\alpha}\partial_\mu\left(\partial_\alpha\lambda_\nu + \partial_\nu\lambda_\alpha\right) - g^{\mu\alpha}\left(\partial_\alpha\lambda_\beta + \partial_\beta\lambda_\alpha\right)\Gamma^{\beta}_{\nu\mu}\,,$$

while the right hand side simplifies to give

$$(\partial_\nu g^{\alpha\beta})\partial_\alpha\lambda_\beta + g^{\alpha\beta}\partial_\nu\partial_\alpha\lambda_\beta.$$

Combining the above two expressions together we find from (28)

$$g^{\mu\alpha}\partial_\mu\partial_\alpha\lambda_\nu = 0\,. \tag{29}$$

As long as $g^{\mu\nu}$ is non-singular general solutions for λ_ν could be found.

Clearly, the system of equations (21), (22) and (28) (or (29)), could admit new solutions that are absent in the Einstein theory. For instance, the Einstein equations with a nonzero cosmological constant (i.e., $T_{\mu\nu} = \Lambda g_{\mu\nu}$), do not admit Minkowski

space as a solution. In contrast with this, the system (21), (22) and (28) is satisfied by the flat space metric

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu = \Lambda\, \eta_{\mu\nu}\,. \tag{30}$$

It is remarkable that one can obtain a flat solution even though the vacuum energy in the Lagrangian is not zero! Similar property exists in unimodular gravity [6] when $T_{\mu\nu} = \Lambda g_{\mu\nu}$, but there are an infinite number of maximally symmetric solutions labeled by the value of a constant curvature. Is this also true in the model (19)? We analyze this issue below. First we notice that the system (21), (22) and (28) admits a de Sitter solution of conventional general relativity (we will focus on the case of a positive cosmological constant only from now on and interesting results can be obtained for both positive and negative Λ, [8]). The conventional dS metric solves (21) with $\lambda_\mu = const.$. What is less obvious is how this solution satisfies (22). To understand this we start with the dS solution in the co-moving coordinate system (this is also applicable to any FRW cosmology)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{31}$$

As it can be checked directly, (31) does not satisfy (22). However, we can define a new time variable

$$\tau \equiv \int \frac{\mathrm{d}t}{a^3(t)}\,, \tag{32}$$

for which the interval becomes

$$\mathrm{d}s^2 = a^6(\tau)\mathrm{d}\tau^2 - a^2(\tau)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{33}$$

This metric satisfies (22) identically. Therefore, a conventional dS metric, or any other spatially-flat FRW cosmology, is a solution of the system (21), (22), (28).

Having established that the flat and conventional dS spaces are two solutions of the theory, let us now look at other possible maximally symmetric solutions. In general, the following ansatz

$$\partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu = c g^{dS}_{\mu\nu}\,, \tag{34}$$

where c is an arbitrary constant, $g^{dS}_{\mu\nu}$ is a dS solution with $R = -4(\Lambda - c)$, does satisfy the equations (21), (22) and (28). However, this is not enough to claim that (34) is a legitimate solution of the theory. This is because equation (34) itself might not be solvable in terms of λ_μ given that $g_{\mu\nu}$ is a dS metric obeying (22); solvability for λ_μ is a necessary condition since it is w.r.t. λ_μ that we varied the action. It is straightforward to check that there is no solution for λ_μ that would satisfy equation (34) if the metric is given by (33). Could there be other forms of dS space that are non-trivially different from (33) and yet satisfy (22)? The answer is no. To see this consider a dS solution in GR in two different coordinate systems. Let us *assume* the opposite, that both of these coordinate systems can be gauge transformed in GR to

two different coordinate systems for which (22) is valid. If true, this would mean that the condition (22) does not completely fix the gauge freedom of GR. On the other hand, we find by performing gauge transformation of (22) that this is only possible if the gauge transformation itself is trivial. Therefore, the form (33) is the unique dS solution which satisfies the constraints (22).

The fact that (34) cannot be a maximally symmetric solution with nonzero $R(g)$ could also be established just by looking at a general expression of the Ricci scalar in terms of the metric $g_{\mu\nu}$ in which the substitution $cg_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu$ is made. If this ansatz could describe a dS space, one could always go to a weak-field regime where the Ricci curvature, as a function of the metric, has to be nonzero. However, the above ansatz for $g_{\mu\nu}$ gives zero R in the leading order of the weak-field approximation.

Summarizing, we conclude that in the class of maximally symmetric spaces there are only two solutions of the theory: (I) The flat space defined in (30); (II) The (A)dS solution as it would appear in conventional GR transformed to a new coordinate system. There could in principle exist other, non-maximally symmetric solutions. It would be interesting to study whether those solutions are physical. If they are one should look for arguments why the maximally symmetric solutions could be preferred in our Universe. On the other hand, if the non-maximally-symmetric solutions are not there, then the model (19) could be a playground for studying the fate of the "old" cosmological constant problem [7]. It is interesting to point out that by integrating out the Lagrange multiplier field one gets a non-local action, with some similarities but also differences of the resulting non-local theory with that of Ref. [10].

We would also point out that the Schwarzschild metric of conventional GR is also a legitimate solution of the CGR. A simplest way to address this is to choose $\lambda_\nu = 0$ and make sure that the known GR solution itself satisfies (22) in a particular coordinate system. Usual solutions of GR can be transformed to satisfy Eq. (22). This is in particular true when $g_{\mu\nu}$ is diagonal and each element of $\sqrt{g} g_{\mu\mu}$ (there is no summation w.r.t. μ here) factorizes into the products of the form $\sqrt{g} g_{\mu\mu} = h(x^\alpha) j(x^\nu) \cdots f(x^\lambda)$, where each function depends on one coordinate only. In this case, the constraints (22) turn into four separate partial differential equations

$$\partial_\mu (\sqrt{g} g_{\mu\mu}^{-1}) = 0, \qquad \mu = 0, 1, 2, 3, \quad \text{no summation w.r.t. } \mu. \tag{35}$$

Suppose we introduce one new coordinate $\tilde{x}^\alpha = \tilde{x}^\alpha(x^\alpha)$ such that it depends only on x^α, and leave all the other coordinates intact. In the new coordinate system

$$\sqrt{-\tilde{g}} = x'^\alpha \sqrt{-g}\,, \tag{36}$$

while

$$\tilde{g}^{\alpha\alpha} = (x'^\alpha)^{-2} g^{\alpha\alpha}\,, \quad \text{and} \quad \tilde{g}^{\mu\mu} = g^{\mu\mu} \ \text{ for } \ \mu \neq \alpha, \tag{37}$$

where we have defined $x'^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tilde{x}^\alpha}$ and chosen it to be positive.

The α-th equation of (35) in the new coordinate system takes the form:

$$\frac{\partial(\sqrt{-\tilde{g}}\tilde{g}^{\alpha\alpha})}{\partial\tilde{x}^{\alpha}} = \frac{\partial[\sqrt{-g}g^{\alpha\alpha}(x'^{\alpha})^{-1}]}{\partial\tilde{x}^{\alpha}} = 0\,. \tag{38}$$

If $\sqrt{-g}g^{\alpha\alpha}$ can be factorized, say as, $\sqrt{-g}g^{\alpha\alpha} = h(x^{\alpha})\psi$ where ψ depends only on coordinates other than x^{α}, we can find the desired $\tilde{x}^{\alpha}$ by simply demanding

$$\frac{h(x^{\alpha})}{x'^{\alpha}(\tilde{x}^{\alpha})} = 1\,, \quad \text{or any constant if more convenient,} \tag{39}$$

and solving this ordinary differential equation. It is not difficult to see that one can carry on the same procedure for each x^{μ}'s without invalidating the previous results, and, therefore, eventually find the new coordinate system that satisfies (22).

The above procedure is directly applicable to the Schwarzschild and FRW solutions. For the latter the result was already given above (see (33)). Here we perform the change of coordinates for the Schwarzschild metric. In a spherically symmetric coordinates

$$\mathrm{d}s^2 = \left(1 - \frac{r_g}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_g}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,. \tag{40}$$

Here, $r_g = 2G_N M$ is the horizon radius of an object. The above described procedure leads to the new coordinate system

$$\begin{aligned} \tilde{r} = r_g \ln\frac{r - r_g}{r} \quad &\Rightarrow \quad r = \frac{r_g}{1 - e^{\tilde{r}/r_g}}\,, \\ \tilde{\theta} = \ln\tan\frac{\theta}{2} \quad &\Rightarrow \quad \theta = 2\tan^{-1} e^{\tilde{\theta}}\,, \end{aligned} \tag{41}$$

in which the Schwarzschild metric becomes

$$\mathrm{d}s^2 = e^{\tilde{r}/r_g}\mathrm{d}t^2 - \frac{e^{\tilde{r}/r_g}}{(e^{\tilde{r}/r_g} - 1)^4}\mathrm{d}\tilde{r}^2 - \frac{r_g^2}{(e^{\tilde{r}/r_g} - 1)^2\cosh^2\tilde{\theta}}\left(\mathrm{d}\tilde{\theta}^2 + \mathrm{d}\varphi^2\right)\,. \tag{42}$$

This metric satisfies (22). The new variable $\tilde{r}$ spans the interval $(-\infty,\ 0]$ as the coordinate r increases from r_g to $+\infty$, while the new angular variable $\tilde{\theta}$ covers the interval $(-\infty,\ +\infty)$. One can also easily describe the interior of the Schwarzschild solution by flipping the sign of the argument of the log in (41).

3.2 Radiative stability

In this section we discuss the issue of quantum loop corrections to (19). In particular, we would like to understand whether this theory is stable w.r.t. the loops. It is clear that quantum-gravitational and matter loops will generate higher dimensional operators entering the action suppressed by the UV cutoff of this theory. This is similar to any theory that is not renormalizable and should be regarded as an

effective field theory below its UV cutoff (for an introduction to an effective field theory treatment of gravity, see, e.g., [11]). However, there is another question of a vital importance in the present context. This is whether loop corrections can generate the potential and/or kinetic terms for the Lagrange multiplier. If this happens, λ_μ cannot be regarded as an auxiliary field and all the results of the previous subsection would be ruined.

We will argue below that this problem is avoided in CGR because of the specific form of (19) which can be completed to a BRST invariant theory. Following Ref. [12], we introduce the variables (as in the previous sections, below we are discussing small fluctuations of the fields)

$$\gamma^{\mu\nu} = \sqrt{g} g^{\mu\nu} , \quad \gamma_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{g}} . \tag{43}$$

It is straightforward to rewrite the Lagrangian (19) in terms of these variables and include the FP ghost term:

$$\mathcal{L} = -\frac{1}{2}\gamma^{\mu\nu}\left(R_{\mu\nu}(\gamma) - \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu\right) + \frac{i}{2}\left(\partial_\mu\bar{c}_\nu + \partial_\nu\bar{c}_\mu\right)\nabla^{\mu\nu}_\alpha c^\alpha . \tag{44}$$

Here the terms in the first parenthesis represent the gravitational part of (19), while the last term introduces the vector-like FP ghost and anti-ghost fields for which the operator $\nabla^{\mu\nu}_\alpha$ is defined as follows:

$$\nabla^{\mu\nu}_\alpha \equiv \gamma^{\mu\tau}\delta^\nu_\alpha\partial_\tau + \gamma^{\nu\tau}\delta^\mu_\alpha\partial_\tau - \partial_\alpha(\gamma^{\mu\nu} \quad) .$$

The important point is that the standard Einstein-Hilbert action can be rewritten in terms of (43) and their first derivatives [12]

$$-\frac{1}{2}\gamma^{\mu\nu}R_{\mu\nu}(\gamma) = -\frac{1}{8}\partial_\rho\gamma^{\mu\tau}\partial_\sigma\gamma^{\lambda\nu}\left(\gamma^{\rho\sigma}\gamma_{\lambda\mu}\gamma_{\tau\nu} - \frac{1}{2}\gamma^{\rho\sigma}\gamma_{\mu\tau}\gamma_{\lambda\nu} - 2\delta^\sigma_\tau\delta^\rho_\lambda\gamma_{\mu\nu}\right) . \tag{45}$$

The FP ghost term in (44) ensures the BRST invariance of this theory [13]. The respective BRST transformations with a continuous Grassmann variable ζ are:

$$\begin{aligned} \delta\gamma^{\mu\nu} &= i\nabla^{\mu\nu}_\alpha c^\alpha\zeta, \\ \delta c^\mu &= ic^\tau\partial_\tau c^\mu\zeta, \\ \delta\bar{c}^\mu = -\lambda_\mu\zeta, &\quad \delta\lambda_\mu = 0 . \end{aligned}$$

Presence of this symmetry ensures that the Lagrange multiplier in (44) does not acquire the kinetic term through the loop corrections. This is because the potentially dangerous term in the effective Lagrangian

$$\partial_\mu\lambda_\nu\langle\gamma^{\mu\nu}(x)\gamma^{\alpha\beta}(0)\rangle\partial_\alpha\lambda_\beta , \tag{46}$$

is zero (up to a total derivatives) due to the transversality of a two point graviton correlation function. The latter can be seen order by order in perturbation theory.

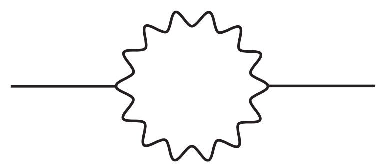

Figure 1: The solid lines denote the λ field (there is no propagator corresponding to these lines, they are depicted to show the vertices); the wave lines correspond to gravitons. The vertices in this one-loop diagram arise due to the cubic interaction of λ with two gravitons originating from the term $\lambda\sqrt{g}$ in the Lagrangian (18).

Let us for simplicity consider this for an expansion about a flat space which is a consistent solution of the theory even if the cosmological constant is present. We introduce the notations $\gamma^{\mu\nu} = \eta^{\mu\nu} - \varphi^{\mu\nu}$, and look at the two-point correlation function of the $\varphi^{\mu\nu}$ field. This correlator has been studied in detail in the conventional approach, in which there is no Lagrange multiplier term in (44), but instead, a standard quadratic gauge fixing term $\frac{(\partial_\mu \gamma^{\mu\nu})^2}{2\zeta}$ with the gauge parameter ζ is introduced. Our theory, on a fixed classical background, corresponds to the limit $\zeta \to 0$. Hence, all the results concerning the quantum loops derived in the conventional approach on a fixed background are also applicable here with the condition that $\zeta \to 0$. The BRST invariance of the theory can be used to deduce the Slavnov-Taylor identities [14] (see, e.g., [15, 16, 4]). The latter guarantee that order by order in perturbation theory the two point correlation function of the $\varphi^{\mu\nu}$ field is transverse (this corresponds to the $\zeta \to 0$ gauge results of Refs. [15, 16, 4])

$$\langle \varphi^{\mu\nu}(x)\varphi^{\alpha\beta}(0)\rangle \propto \Pi^{\mu\alpha}\Pi^{\nu\beta} + \Pi^{\mu\beta}\Pi^{\nu\alpha} - \Pi^{\mu\nu}\Pi^{\alpha\beta},$$
$$\text{where} \qquad \Pi^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}\,.$$

In conclusion, the presence or absence of the FP ghosts does not affect the classical equations of motion, and for the classical analysis it is acceptable to ignore them and study the Lagrangian (19). The only quantum mechanically consistent theory is that with the FP ghosts described by (44) and all the classical results obtained above hold in this theory. Furthermore those results are stable w.r.t. the quantum loop corrections.

We would like to comment on a similar issue in the context of UGR [6]. The Lagrangian of this theory, as given in (18), is likely to generate quadratic terms for λ via loops. To see this we start with a one-loop diagram of Fig. 1. This diagram is logarithmically divergent and will generate an additional term proportional to

$$\frac{\lambda^2}{M_{\rm Pl}^4}\log\left(\frac{\mu_{UV}}{\mu_{IR}}\right)\,, \tag{47}$$

where μ_{UV} and μ_{IR} denote the UV and IR scales respectively, and we have restored $M_{\rm Pl}^2$ in front of the $\sqrt{g}R$ which resulted in the $1/M_{\rm Pl}^4$ coefficient in (47). Naively, the

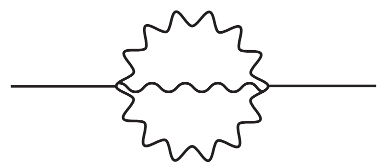

Figure 2: The solid lines denote the λ field (there is no propagator corresponding to these lines, they are depicted to show the vertices); the wave lines correspond to gravitons. The vertices in this one-loop diagram arise due to the quartic interaction of λ with three gravitons originating from the term $\lambda\sqrt{g}$ in the Lagrangian (18).

above term may seem to be irrelevant because of the $1/M_{\rm Pl}^4$ suppression. However, to get a right scaling we should restore the canonical dimensionality of λ. This is achieved by substitution $\lambda \to M_{\rm Pl}^3\alpha$, after which the α-dependent terms in the effective Lagrangian take the form

$$M_{\rm Pl}^3\alpha(\sqrt{g}-1) + M_{\rm Pl}^2\alpha^2\log\left(\frac{\mu_{UV}}{\mu_{IR}}\right). \tag{48}$$

Because of the new induced term, the field α acquires a Planckian mass and ceases to be an auxiliary field. Furthermore, we could also look at a two-loop diagram of Fig 2. A simple power-counting of the momenta running in the loops shows that this diagram generates not only a mass term for α but also its kinetic term:

$$\frac{(\partial_\mu\lambda)^2}{M_{\rm Pl}^6}\log\left(\frac{\mu_{UV}}{\mu_{IR}}\right) \to (\partial_\mu\alpha)^2\log\left(\frac{\mu_{UV}}{\mu_{IR}}\right).$$

Higher loops are also expected to generate similar terms, and the above arguments are hard to avoid unless the theory (18) is amended by new interactions. Perhaps the BRST invariant completion of UGR proposed in Ref. [17] can cure this; it would be interesting to perform explicit calculations in the framework of [17] to see whether the radiative stability is restored. Since the model of [17] is BRST invariant one would expect a positive outcome. However, we should point out that the bosonic part of [17] contains additional gauge fixing terms needed to completely restrict parametrization invariance of the theory, and, from this perspective, it differs from UGR.

Finally, we would like to calculate a response of the graviton field to a source. In the linearized approximation the number of physical propagating degrees of freedom of CGR should be the same as in GR. In the linearized approximation $\partial_\mu\varphi^{\mu\nu}=0$ due to (22), and $\partial^2\lambda_\nu=0$ due to (28). Then, the equation (21) simplifies to

$$\Box\varphi_{\mu\nu} = T_{\mu\nu}\,. \tag{49}$$

The above equation is identical to an expression for the response in the Einstein theory. This of course is a consequence of the fact that the free propagator of (44), coincides with the graviton propagator of GR in the harmonic gauge $\partial_\mu\varphi^{\mu\nu} = -\partial_\mu h^{\mu\nu} + \frac{1}{2}\partial^\nu h = 0$, where $g^{\mu\nu}\simeq\eta^{\mu\nu}-h^{\mu\nu}$.

3.3 Comments on massive theories

It is difficult to construct a consistent Lorentz-invariant nonlinear model of a massive graviton propagating on a Minkowski background. In the linearized approximation the only consistent massive deformation of GR is the Fierz-Pauli (F-P) model [18]

$$\mathcal{L} = -\frac{\sqrt{g}}{2}R + \frac{\sqrt{g}}{4}m_g^2(h_{\mu\nu}h^{\mu\nu} - h^2)\,, \tag{50}$$

were $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. In the quadratic approximation this model describes a massive spin-2 state with 5 degrees of freedom. However, a nonlinear completion of this theory is not unique, and so far there is no known non-linear theory in four-dimensions that would be consistent. A rather general class of nonlinear completions of the F-P theory give rise to unbounded from below Hamiltonian [19]. This manifests itself in classical instabilities for which the time scale can be substantially shorter than the scale of the inverse graviton mass [20, 21, 22].

The reason for this instability is that at the nonlinear level a ghost-like sixth "degree of freedom" shows up. This should have been expected because of the following. Ten degrees of freedom of $g_{\mu\nu}$ in (50) are restricted only by four independent Bianchi identities. Hence, six degrees of freedom should remain. The absence of the sixth degree of freedom in the linearized theory was just an artifact of the linearized approximation itself [19].

It is interesting to ask the following question: what if we start with a mass deformation of a constrained gravitational theory instead of modifying GR as in (50)? Straightforward calculations show that the mass deformation of unimodular gravity (18) or the constrained gravity (19) leads to a theory with a ghost in the linearized approximation. This ghost can be removed, at least in the linearized theory, if one considers a mass deformation of a model with both constraints $\sqrt{g} = 1$ and $\partial_\mu\sqrt{g}g^{\mu\nu} = 0$. Because this set of equations imposes 5 conditions on ten components of $g_{\mu\nu}$, one should expect this theory to propagate 5 physical degrees of freedom. Below we will discuss the advantages as well as difficulties of this approach.

The Lagrangian of the massive "hybrid model" that combines the two constraints mentioned above takes the form:

$$\mathcal{L} = -\frac{\sqrt{g}}{2}R + \frac{\sqrt{g}}{4}m_g^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \sqrt{g}g^{\mu\nu}\partial_\mu\lambda_\nu - \lambda(\sqrt{g} - 1)\,. \tag{51}$$

Variation of the action w.r.t.the Lagrange multipliers λ and λ_μ yields the constraints:

$$\sqrt{g} = 1; \qquad \partial_\mu\sqrt{g}g^{\mu\nu} = 0\,. \tag{52}$$

We now turn to the linearized approximation about a flat space to study a graviton propagator. In this approximation the constraints (52) reduce to $h = 0$, and $\partial^\mu h_{\mu\nu} = 0$ respectively. The equation of motion becomes

$$G_{\mu\nu} - m_g^2 h_{\mu\nu} - \partial_{(\mu}\lambda_{\nu)} + (\partial^\alpha\lambda_\alpha)\eta_{\mu\nu} - \lambda\eta_{\mu\nu} = T_{\mu\nu}. \tag{53}$$

The trace equation and the Bianchi identities give respectively

$$-4\lambda + 2\partial^\mu \lambda_\mu = T\,, \quad \partial_\mu \lambda + \partial^2 \lambda_\mu = 0\,,$$

which can be solved to obtain

$$\lambda_\mu = \frac{\partial_\mu T}{6\Box}\,, \qquad \lambda = -\frac{T}{6}\,. \tag{54}$$

Substituting these solutions into (53) we find

$$h_{\mu\nu} = \frac{T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu}}{\Box + m_g^2} + \frac{\partial_\mu \partial_\nu T}{3\Box(\Box + m_g^2)}\,. \tag{55}$$

From the tensorial structure of (55) we conclude that the theory propagates five physical polarizations, as it should. Moreover, unlike F-P gravity, the propagator (55) has a well-defined $m_g \to 0$ limit. This is similar to the soft behavior of massive gauge field propagator discussed in Appendix A and B, and to "softly massive gravity" emerging in higher dimensional constructions [23, 24].

Could (51) be a consistent model of a massive graviton in 4D? There still is a long way to go in order to find out whether (51) is a theoretically sound theory. There are three major checks one should perform.

(i) The main problem of the F-P gravity stems from the fact that the Hamiltonian of the nonlinear theory is unbounded below. Hence, one should understand whether the same problem is evaded by the hybrid model (51). We studied this question partially and have shown that the unbounded terms that appear in the F-P massive gravity do not arise in (51). To understand this we look at the ADM decomposition of the metric

$$g_{\mu\nu} = \begin{pmatrix} N^2 - \tilde\gamma_{ij} N^i N^j & -N^j \tilde\gamma_{ij} \\ -N^j \tilde\gamma_{ij} & -\tilde\gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & -\tilde\gamma^{ij} + \frac{N^i N^j}{N^2} \end{pmatrix}.$$

In terms of the new variables the Einstein-Hilbert Lagrangian takes the form

$$-\frac{\sqrt{\tilde\gamma}}{2} N (R^{(3)} + K_{ij} K^{ij} - K^2)\,, \tag{56}$$

where $\tilde\gamma = \det \tilde\gamma_{ij}$, $R^{(3)}$ is the 3-dimension Ricci curvature calculated with the metric $\tilde\gamma_{ij}$ and the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2N}(\dot{\tilde\gamma}_{ij} - D_i N_j - D_j N_i),$$

contains a covariant derivative D_i with respect to the metric $\tilde\gamma_{ij}$. The problem of the F-P gravity arises because the lapse N acquires a quadratic term in non-linear realizations of F-P gravity. Hence, it ceases to be a Lagrange multiplier and does not restrict the propagation of an extra sixth "degree of freedom" which is ghost-like.

In the hybrid model, however, because of $\sqrt{g} = 1$ we get that $N = 1/\sqrt{|det\tilde{\gamma}_{\mu\nu}|}$. This constraint enables one to remove the sixth degree of freedom and the dangerous terms that previously led to unboundedness of the F-P Hamiltonian [19]. Regretfully, the expression for the Hamiltonian of the hybrid model is rather complicated and it is difficult to see that there are no other sources of rapid classical instability there.

(ii) Even if the rapid classical instabilities are removed, the main question is whether the BRST invariant completion of (51) exists. The BRST symmetry, could be a guiding principle determining a unique nonlinear completion of (51)(or any other massive theory), which at this stage is completely arbitrary. One could hope that the BRST and ghost charges can be used to define the Hilbert space of physical states of the theory so that even if the Hamiltonian is not bounded below, the states of negative energy do not appear in the final states (i.e., they are projected out by the conditions $Q_{\mathrm{BRST}}|\mathrm{Phys}\rangle = Q_{\mathrm{Ghost}}|\mathrm{Phys}\rangle = 0$ and cannot be emitted in any process). At the moment it is not clear whether such a construction is possible, but we plan to return to this set of questions in future.

(iii) The question of radiative stability of (51) is something one should worry about. In general the Lagrange multipliers of the massive theory will acquire the mass and kinetic terms, and this would lead to propagation of a new degree of freedom. Only hope here could be to complete (51) in a BRST invariant way so that the resulting theory does not generate the quadratic and higher terms for the Lagrange multipliers.

Even if all the above three issues (i-iii) are positively resolved, one needs to amend the massive model to make it consistent with the data. The point is that the scalar polarization of a massive graviton couples to sources and gives rise to contradictions with the Solar system data. In the model of Ref. [25] the similar problems is solved due to nonlinear effects that screen the undesirable scalar polarization at observable distances [26, 27, 28, 29]. In the present model, at least naively, such a mechanism does not seem to be operative, and some new ideas are needed.

Acknowledgments

We thank Savas Dimopoulos, Gia Dvali, Andrei Gruzinov, Nemanja Kaloper, Matthew Kleban, Massimo Porrati, Adam Schwimmer, Tom Taylor, Jay Wacker and Dan Zwanziger for useful discussions and comments. The work was supported by NASA grant NNG05GH34G. GG is also partly supported by NSF grant PHY-0403005, and YS by graduate student funds provided by New York University. YS would also like to thank Andrei Gruzinov for his support through David and Lucile Packard Foundation Fellowship.

Appendix A: Qantization of CQED: Stückelberg formalism

There is another way of quantizing (3). We can restore the gauge invariance of (3) using the Stückelberg method and then follow the standard FP procedure of fixing the gauge and introducing the FP ghosts. Let us discuss this in some detail.

We start by rewriting (3) as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \lambda(\partial^\mu B_\mu - \Box\varphi) + J^\mu(B_\mu - \partial_\mu\varphi), \tag{57}$$

here we have introduced the notations:

$$B_\mu = A_\mu + \partial_\mu\varphi \quad \text{and} \quad F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{58}$$

A Lagrangian, similar to (57), in a context of a theory with a non-conserved current was recently discussed in [30]. The Lagrangian (57) is invariant under the following gauge transformation

$$\delta B_\mu = \partial_\mu\alpha \quad \text{and} \quad \delta\varphi = \alpha, \tag{59}$$

where α is an arbitrary function. To fix this freedom we choose a gauge similar to the R_ξ-gauge that eliminates a mixing terms between λ and B. This can be achieved by adding into the Lagrangian the following gauge fixing term:

$$\Delta\mathcal{L}_{GF} = \frac{1}{2\xi}(\partial^\mu B_\mu - \xi\lambda)^2. \tag{60}$$

Here, ξ is an arbitrary gauge parameter. Furthermore, it is easy to find the FP determinant and introduce the FP ghosts into the theory. The total Lagrangian that includes the gauge fixing and FP ghost terms reads:

$$\mathcal{L}_{\rm tot} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial^\mu B_\mu)^2 - i\bar{c}\Box c - \lambda\Box\varphi + \frac{\xi}{2}\lambda^2 + J^\mu(B_\mu - \partial_\mu\varphi). \tag{61}$$

The first three terms on the right hand side of the above expression constitute a free Lagrangian of gauge-fixed QED. In addition there are other states in (61). To understand their nature, we integrate out from (61) the auxiliary field λ. As a result, the terms of (61) containing λ get replaced as

$$-\lambda\Box\varphi + \frac{\xi}{2}\lambda^2 \to -\frac{1}{2\xi}(\Box\varphi)^2\,. \tag{62}$$

Because the propagator of φ is $\frac{\xi}{\Box^2} = lim_{\gamma\to 0}\frac{1}{\gamma}(\frac{\xi}{\Box} - \frac{\xi}{\Box+\gamma})$, it is more appropriate to think of two states, described by (62), one of which has a positive-sign kinetic term and the other one is ghost-like [5]. However, these field are not present in the

[5]This could also be understood directly from (61) by making substitutions: $\lambda = a+b$, $\varphi = a-b$, which generate two kinetic terms one for a with a right sign and another one for b with a wrong sign. These two fields will also have ξ-dependent masses and mass mixing that originate from the $\xi\lambda^2$ term.

physical on-shell spectrum of the theory. In the limit $\xi \to 0$ the field φ is frozen, the Lorentz-gauge fixing condition, $\partial^\mu B_\mu = 0$, is enforced, and the on-shell spectrum consists of two physical polarizations of a photon. Since physics cannot depend on a choice of the value of the gauge parameter ξ, the model (61) propagates two on-shell polarizations. Off-shell, however, there are four propagating degrees of freedom. The two non-physical degrees of freedom are longitudinal and time-like components of a photon, while λ plays a role of the canonically conjugate momentum to A_0 (alternatively, if λ is adopted as a canonical coordinate A_0 becomes its conjugate momentum).

It is straightforward to read off the propagators from (61)

$$\Delta_B^{\mu\nu} = -\frac{\eta^{\mu\nu} - (1+\xi)\partial^\mu\partial^\nu/\Box}{\Box} \quad , \qquad \Delta_\varphi \partial^\mu \partial^\nu = -\frac{\xi \partial^\mu \partial^\nu}{\Box^2} \, . \tag{63}$$

Adding these two contributions together we find that the dependence on the gauge parameter ξ cancels out and we obtain (16). This confirms our previous conclusion that the φ field is a gauge artifact. It is also instructive to rewrite the Lagrangian (61) in terms of the original field A_μ. The latter looks as follows:

$$\mathcal{L}_{\rm tot} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 - \frac{1}{\xi}(\partial^\mu A_\mu)\Box\varphi - i\bar{c}\Box c + A_\mu J^\mu \, , \tag{64}$$

where we have integrated out the λ field. Here we see a difference from conventional QED. The additional ξ dependent term ensures that the resulting free propagator, for arbitrary values of ξ, coincides with the Landau gauge propagator of QED. This is due to an extra field φ which has no kinetic term but acquires one through the kinetic mixing with the gauge field (64). The mixing term itself is gauge-parameter dependent and this is what cancels the ξ dependence of the QED propagator.

A.1. On massive deformation of CQED

In this subsection we study the massive deformation of the Lagrangian (3). We expect, because of the classical constraint, the massive theory to be different off-shell from the conventional massive electrodynamics (the Proca theory).

To proceed, we add to the Lagrangian (3) the following mass term:

$$\Delta l = \frac{1}{2} m_\gamma^2 A_\mu^2 \, . \tag{65}$$

One can think of this term as arising from some higher-dimensional operator in which certain fields acquiring VEV's generate (65) while these fields themselves become heavy and decouple from the low-energy theory.

It is straightforward to see that Eq. (4) gets modified by the mass term

$$\partial^\nu F_{\nu\mu} + m_\gamma A_\mu + J_\mu - \partial_\mu \lambda = 0 \, , \tag{66}$$

while the constraint $\partial_\mu A^\mu = 0$ remains intact. Taking a derivative of (66) one obtains the equation of motion for λ

$$\Box\,\lambda = \partial^\mu J_\mu. \tag{67}$$

From the above we find a solution with a nonzero constant background current $\partial_\mu \lambda = C_\mu$ which is identical to that of the constrained massless theory.

It is interesting that the constrained massive theory also has a continuous BRST invariance if the FP ghost fields are introduced. Indeed, consider the Lagrangian

$$l = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda(\partial_\mu A^\mu) + \frac{1}{2} m_\gamma^2 A_\mu^2 - i\bar{c}\Box c. \tag{68}$$

As before, the presence of the FP ghost in (68) does not affect the discussions of the classical equations. On the other hand, due to these fields the Lagrangian (68) becomes invariant under the following continuous BRST transformations:

$$\delta A_\mu = i\zeta \partial_\mu c\,, \quad \delta c = 0, \quad \delta\bar{c} = \lambda\zeta\,, \quad \delta\lambda = i m_\gamma^2 \zeta c\,, \tag{69}$$

where ζ is a continuous Grassmann parameter. Notice that λ transforms to compensate for the non-invariance of the mass term[6]. Moreover, $\delta^2 A_\mu = \delta^2 c = \delta^2 \lambda = 0$, while $\delta^2 \bar{c} \neq 0$, but $\delta^3 \bar{c} = 0$.

Let us now turn to the discussion of the spectrum of this theory. First we evaluate the propagator of the gauge field. For this we follow the method used in section 2. The result is:

$$\Delta^{\mu\nu} = -\lim_{\gamma\to 0}\left[\eta^{\mu\nu} - \frac{1+\gamma}{\Box - \gamma m_\gamma^2}\partial^\mu\partial^\nu\right]\frac{1}{\Box + m_\gamma^2} = -\frac{\eta^{\mu\nu} - \partial^\mu\partial^\nu/\Box}{\Box + m_\gamma^2}. \tag{70}$$

This should be contrasted with the propagator of conventional massive QED (the Proca theory):

$$\Delta^{\mu\nu}_{\rm Proca} = -\frac{\eta^{\mu\nu} + \partial^\mu\partial^\nu/m_\gamma^2}{\Box + m_\gamma^2}. \tag{71}$$

The key difference of (70) from (71) is the absence in (70) of the longitudinal term that is inversely proportional to the mass square. Because of this, the propagator (70) does have a good UV behavior, while the propagator (71) does not. In an Abelian theory with a conserved current this hardly matters since the longitudinal parts of the propagators do not contribute to physical amplitudes. However, this could become important for non-Abelian and gravitational theories where the matter currents are only covariantly conserved, so we carry on with this discussion.

[6]The above transformations (69) differ from the standard BRST transformations of a massive Abelian theory in which massive FP ghost need to be introduced. We emphasize here the transformations (69) because they will be straightforwardly generalized to the non-Abelian massive case in the next section.

A natural question that arises is what is the mechanism that softens the UV behavior of (70) as compared to (71)? Usually in massive gauge theories this is achieved by introducing a Higgs field that regulates the UV behavior of (71). Therefore, on top of the three physical polarizations of a massive gauge field, there should be a new state that replaces the role of the Higgs. To see this state manifestly we rewrite (70) as follows:

$$\Delta^{\mu\nu} = -\frac{\eta^{\mu\nu} + \partial^\mu\partial^\nu/m_\gamma^2}{\Box + m_\gamma^2} + \frac{\partial^\mu\partial^\nu}{m_\gamma^2\Box}\,. \tag{72}$$

The first term on the r.h.s. is just the Proca propagator of three massive polarizations; the additional term represents a new massless derivatively-coupled degree of freedom. To uncover the nature of this extra state we rewrite the Lagrangian of the constrained massive theory as follows:

$$l = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_\gamma^2\left(A_\mu - \frac{\partial_\mu\lambda}{m_\gamma^2}\right)^2 - \frac{(\partial_\mu\lambda)^2}{2m_\gamma^2} - i\bar{c}\Box c + J_\mu A^\mu\,. \tag{73}$$

Notice that the quantity in the parenthesis is invariant under the BRST transformations (69), while the non-invariance of the λ-kinetic term under (69) is compensated by the terms coming from the FP ghost kinetic term. Defining a field

$$B_\mu \equiv A_\mu - \frac{\partial_\mu\lambda}{m_\gamma^2}\,, \tag{74}$$

we end up with the following theory:

$$l = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_\gamma^2 B_\mu^2 + B_\mu J^\mu - \frac{1}{2m_\gamma^2}(\partial_\mu\lambda)^2 - i\bar{c}\Box c + \frac{\partial_\mu\lambda}{m_\gamma^2}J^\mu. \tag{75}$$

The first three terms on the r.h.s. of the above expression represent the Proca Lagrangian of massive electrodynamics. There are also additional terms in (75). These are the λ kinetic term and FP ghost kinetic term. These two terms form a sector of the theory that is invariant under the continuous BRST transformations $\delta\lambda = im_\gamma^2\zeta c$, $\delta\bar{c} = \zeta\lambda$ and $\delta c = 0$. This symmetry is an exact one if the current J_μ is either conserved, as it is in an Abelian theory, or transforms w.r.t. BRST in an appropriate way as it will for a non-Abelian theory discussed in the next section. The λ kinetic term in (75) has a wrong sign, and this state is ghost-like. While J_μ is conserved, λ is decoupled from the rest of the physics and can be ignored for all the practical purposes. Nevertheless, it is interesting to understand whether the state λ could belong to a Hilbert space of physical states. This space could be defined by introducing the BRST and Ghost charges and postulating $Q^{\rm BRST}|{\rm Phys}>= 0$ and $Q^{\rm Ghost}|{\rm Phys}>= 0$. The above conditions could ensure that quanta of the λ field do not belong to the set of physical *in* or *out* states of the theory. However, as we discussed previously, the BRST transformations (69) are rather peculiar and the construction of the Hilbert space of physical states could differ from the conventional one.

Here we discuss the properties of the physical degrees of freedom in the Hamiltonian formalism. Let us ignore the external current J_μ for the time being. The canonical momenta conjugate to A_μ read

$$\pi_\mu \equiv \frac{\partial l}{\partial \dot{A}^\mu} = -F_{0\mu} + \lambda \delta_{0\mu} \, . \tag{76}$$

The key difference of this from QED is that (3) contains time derivative of A_0, and, as a result, the primary constraint of QED, $\pi_0 = 0$, is replaced by the relation

$$\pi_0 = \lambda. \tag{77}$$

Thus, λ is just a conjugate momentum for A^0. The above expression can be used to determine λ, and if so, it does not constrain π_0. Excluding λ by (77), the extended Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}\pi_j^2 \, + \, \frac{1}{2}(\epsilon_{ijk}\partial_j A_k)^2 + \pi_0 \partial_j A_j + A_0 \left(\partial_j \pi_j\right) \, . \tag{78}$$

To study this system further we look at the two equations governing the time evolutions of π_0 and A_0 as

$$\dot{\pi}_0 \, = \, \{\pi_0 \, \mathcal{H}\} \, , \qquad \dot{A}_0 \, = \, \{A_0 \, \mathcal{H}\} \, , \tag{79}$$

where $\{..\}$ denotes the canonical Poisson brackets. This reduces to the following relations

$$\dot{\pi}_0 + \partial_i \pi_i = 0 \, , \qquad \partial_\mu \, A^\mu = 0 \, , \tag{80}$$

both of which were already given in a different form by equations of motion derived in section 2. Furthermore, requiring that the time derivatives of these two expressions are identically zero

$$\{\dot{\pi}_0 + \partial_i \pi_i, \mathcal{H}\} = 0 \, , \quad \{\partial_\mu A^\mu, \mathcal{H}\} = 0, \tag{81}$$

we find the following two additional equations

$$\Box \, \pi_0 = 0 \, , \qquad \Box \, A^0 + \partial_i \pi_i = 0. \tag{82}$$

Further time derivatives are identically satisfied.

From the first equation of (82) we find that π_0 can either be a plane wave or a trivial harmonic function (or a superposition of the two). The second equation of (82), however, dictates that π_0 can only assume the trivial solutions. To show this suppose that the solution for π_0 is a plane wave. Then, according to

$$\Box A^0 = -\partial_i \pi_i = \dot{\pi}_0, \tag{83}$$

A^0 cannot have a nonsingular solution because of the mass-shell condition. Therefore, to avoid non-physical solutions a trivial solution for π_0 should be taken. As a result, equation (80) and (82) together remove π_0 and A^0 from the list of free variables and impose one condition on π_i's and one on A^i's, thus, reducing the number of physical degrees of freedom to two. This is consistent with the Nakanishi-Lautrup condition that prescribes to physical states to satisfy $\lambda^{(-)}|\mathrm{Phys}\rangle = \pi_0^{(-)}|\mathrm{Phys}\rangle = 0$.

In the massive case the Hamiltonian density is modified to be

$$\mathcal{H} = \frac{1}{2}\pi_j^2 + \frac{1}{2}(\epsilon_{ijk}\partial_j A_k)^2 + \pi_0 \partial_j A_j + A_0\,(\partial_j \pi_j) + \frac{1}{2}m_\gamma^2(A_i^2 - A_0^2)\,. \tag{84}$$

Time derivatives of π_0 and A_0 give rise to the following relations

$$\dot{\pi}_0 + \partial_i \pi_i = m_\gamma^2 A_0\,, \qquad \partial_\mu\, A^\mu = 0\,. \tag{85}$$

Their time derivatives lead to Eqs. (82). The above four equations, however, are no longer enough to remove two extra degrees of freedom. Indeed,

$$\Box\, A^0 + m_\gamma^2 A^0 = -\partial_i \pi_i = \dot{\pi}_0, \tag{86}$$

and since $m_\gamma \neq 0$, even if π_0 were a plane wave solution of $\Box\,\pi_0 = 0$, non-singular solutions A^0 can be obtained. Therefore $\partial_i \pi_i$ is no longer constrained to zero, as it was in the massless case. Hence, such a theory propagates three degrees of freedom.

Furthermore, using the relations (85) one can rewrite the Hamiltonian as follows

$$\mathcal{H} = \frac{1}{2}\pi_j^2 + \frac{1}{2}(\epsilon_{ijk}\partial_j A_k)^2 + \frac{1}{2}m_\gamma^2\left(A_i - \frac{\partial_i \pi_0}{m_\gamma^2}\right)^2 - \frac{(\partial_i \pi_0)^2}{2m_\gamma^2} - \frac{\dot{\pi}_0^2}{2m_\gamma^2} + \frac{(\partial_i \pi_i)^2}{2m_\gamma^2}. \tag{87}$$

Defining a new field

$$B_i = A_i - \frac{\partial_i \pi_0}{m_\gamma^2}, \tag{88}$$

we find the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\pi_j^2 + \frac{(\partial_i \pi_i)^2}{2m_\gamma^2} + \frac{1}{2}(\epsilon_{ijk}\partial_j B_k)^2 + \frac{1}{2}m_\gamma^2 B_i^2 - \frac{1}{2m_\gamma^2}[\pi_\lambda^2 + (\partial_i \lambda)^2], \tag{89}$$

where the pairs of canonical coordinates and their conjugate momenta are B_i and π_i and λ and π_λ. The λ field makes a negative contribution to the energy density. However, this field is decoupled from all the sources and, thus, cannot be produced to grow the negative energy.

Appendix B: Constrained non-Abelian gauge fields

Here we generalize the constrained approach to the massless and massive non-Abelian fields. To understand the essence of these models we start with the discussion of the massless case. The Lagrangian reads as follows:

$$l = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + A^a_\mu J^{a\mu} + \lambda^a (\partial_\mu A^\mu)^a \,. \tag{90}$$

Equations of motion for the gauge fields and Lagrange multiplier are respectively:

$$D^\nu F^a_{\nu\mu} + J^a_\mu - \partial_\mu \lambda^a = 0\,, \tag{91}$$

$$\partial_\mu A^{a\mu} \;=\; 0\,. \tag{92}$$

An equation for λ follows by taking a covariant derivative of (91)

$$D^\mu \partial_\mu \lambda^a = D^\mu J^a_\mu \,. \tag{93}$$

In a theory with a covariantly conserved source there could exist new non-trivial classical solutions for λ such that $\partial_\mu \lambda$ is also covariantly conserved. One particular solution is a path-ordered Wilson line

$$\partial_\mu \lambda(x) = \mathcal{P}\mathrm{exp}\left(-ig \int_y^x A_\mu(z) dz^\mu\right) \partial_\mu \lambda(y)\,. \tag{94}$$

Thus, if there is a non-zero external colored current solution $\partial_\mu \lambda$ at some point, then its value at any other point can be calculated according to (94).

The next issue to be address is that of quantum consistency of this approach. Again an important fact is that the Lagrangian (90) can be completed to a BRST invariant form:

$$l = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + A^a_\mu J^{a\mu} + \lambda^a (\partial^\mu A^a_\mu) \, - i\bar{c}^a \partial^\mu D^{ab}_\mu c^b \,. \tag{95}$$

Since at the classical solutions that we consider the FP ghost fields vanish, (95) recovers all the classical results of (90). The explicit BRST transformations leaving (95) invariant are:

$$\delta A^a_\mu = i\zeta D^{ab}_\mu c^b \,, \quad \delta c^a = -\frac{i}{2} g\zeta f^{abd} c^b c^d \,, \quad \delta \bar{c}^a = \lambda^a \zeta \,, \quad \delta\lambda^a = 0\,. \tag{96}$$

The expression for the BRST and Ghost currents, as well as the subsidiary conditions that guarantee the unitarity and completeness of the physical Hilbert space of states are the standard ones [4].

As in the case of a photon, the massless free propagator reads:

$$\Delta^{ab}_{\mu\nu} = -\delta^{ab} \, \frac{\eta_{\mu\nu} - \partial_\mu \partial_\nu / \Box}{\Box} \,. \tag{97}$$

Finally, due to the presence of the BRST symmetry the Lagrange multiplier does not acquire kinetic term via the loop corrections. This is because the two-point correlation functions of the A_μ field is guarantied to be transverse due to the presence of the FP ghost and the transverse structure of the tree-level propagator (97)

$$\partial^\mu\lambda^a \, \langle A_\mu^a \, A_\nu^b\rangle\partial^\nu\lambda^b \sim \partial^\mu\lambda^a\delta^{ab}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial^\nu\lambda^b = 0 \,, \tag{98}$$

to all orders of perturbation theory.

B.1. Stückelberg formalism

Instead of using BRST symmetry to arrive from (90) to (95), here we follow a conventional route. First we restore gauge symmetry of (90) and then fix that restored gauge invariance and introduce the appropriate FP ghosts. Let us start by defining new variables:

$$igA_\mu = U^+D_\mu U, \quad \text{where} \quad U = e^{it^a\pi^a}, \qquad D_\mu = \partial_\mu + igB_\mu \,, \tag{99}$$

and rewrite the Lagrangian (90) in the following form

$$l = \frac{1}{2}\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda\partial^\mu[U^+D_\mu U]}{ig} + \frac{U^+D_\mu U}{ig}J^\mu\right) . \tag{100}$$

The above Lagrangian is gauge invariant under the local transformations of $U \to e^{i\alpha^a t^a}U$ and $B_\mu \to e^{i\alpha^a t^a}B_\mu e^{-i\alpha^a t^a} + \frac{i}{g}[\partial_\mu e^{i\alpha^a t^a}]e^{-i\alpha^a t^a}$, where t^a denote the generators of a local gauge group, and α^a's are gauge transformation functions[7]. We can proceed further and fix this gauge freedom by introducing into the Lagrangian the following term

$$\frac{1}{4\xi}\mathrm{Tr}(\partial_\mu B^\mu - \xi\lambda)^2 \,. \tag{101}$$

This term removes quadratic mixing between B_μ and λ, and after integrating out λ and rewriting it back in terms of the A_μ field, the result reads as follows:

$$l = \frac{1}{2}\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{\xi}(\partial_\mu A^\mu)(\partial^\mu\Pi_\mu) - i\bar{c}\partial_\mu D_\mu c + A_\mu J^\mu\right), \tag{102}$$

where $\Pi_\mu = UA_\mu U^+ + \frac{i}{g}[\partial_\mu U]U^+ - A_\mu$. All the terms of the Lagrangian (102), except the third one, is what one gets in the conventional approach. The third, gauge dependent term, has a structure that guarantees that the free propagator for an arbitrary value of ξ coincides with the Landau gauge propagator of the conventional approach. This is due to the extra fields π^a's which by themselves have no kinetic term but acquire one through the gauge-parameter-dependent kinetic mixing term with A_μ in (102).

[7]Notice that the current J_μ and the Lagrange multiplier λ are not supposed to transform under these gauge transformations. This can be achieved by rewriting the fundamental fields out of which J_μ is constructed (as well as rewriting λ) in terms of new fields rescaled by U's. Under the gauge transformations the new field transform in a conventional way but their variance is compensated by transformations of U's, so that J_μ and λ stay invariant.

B.2. Comments on massive constrained non-Abelian fields

Let us start with a component form of the constrained massive Lagrangian

$$l = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + A^a_\mu J^{a\mu} + \lambda^a(\partial_\mu A^\mu)^a + \frac{1}{2}M^2 A^a_\mu A^a_\mu - i\bar{c}^a\partial^\mu D^{ab}_\mu c^b\,. \quad (103)$$

It is interesting that the action of this theory is BRST invariant under the following transformations (the Lagrangian transforms as a total derivative)

$$\delta A^a_\mu = i\zeta D^{ab}_\mu c^b\,, \quad \delta c^a = -\frac{i}{2}g\zeta f^{abd}c^b c^d\,, \quad \delta\bar{c}^a = \lambda^a\zeta\,, \quad \delta\lambda^a = iM^2\zeta c^a\,. \quad (104)$$

Notice that the Lagrange multiplier does transforms w.r.t. the BRST and compensates for the non-invariance of the mass term. The above BRST transformations have the following peculiar properties:

$$\delta^2 A^a_\mu = \delta^2 c^a = 0; \quad \delta^2\bar{c}^a \neq 0 \neq \delta^2\lambda^a; \quad \delta^4\bar{c}^a = \delta^3\lambda^a = 0\,. \quad (105)$$

The respective BRST and Ghost currents (the latter is due to the ghost-rescaling symmetry) $\bar{c} \to e^\alpha\bar{c},\; c \to e^{-\alpha}c$) can be derived

$$J^{\rm BRST}_\mu = -F_{\mu\nu}D_\nu c + \lambda D_\mu c - \frac{ig}{2}\partial_\mu(\bar{c}^a)f^{abd}c^b c^d, \quad J^{\rm Ghost}_\mu = i(\bar{c}^a D_\mu c^a - (\partial_\mu\bar{c}^a)c^a) \quad (106)$$

The physical states could be defined in analogy with the standard approach

$$Q_{\rm BRST}|{\rm Phys}>= 0, \quad Q_{\rm Ghost}|{\rm Phys}>= 0\,, \quad (107)$$

where the BRST and Ghost charges are defined as $Q_{\rm BRST} = \int d^3x J^{\rm BRST}_0(t,x)$ and $Q_{\rm Ghost} = \int d^3x J^{\rm Ghost}_0(t,x)$. This suggest that the λ state, which upon the diagonalization of the Lagrangian acquires a ghost-like kinetic term, should not be a part of the physical Hilbert space of *in* and *out* states of the theory. The fictitious λ particle should be allowed to propagate as an intermediate state in Feynman diagrams softening the UV behavior of the theory, however, it cannot be emitted as a final *in* or *out* state of the theory. In this respect it should be similar to a FP ghost. However, because the peculiar properties of the above BRST transformations (105), it still remains to be seen that for a rigorous construction of a positive semi-definite norm Hilbert space of states with unitary S-matrix elements the conditions (107) are enough. Besides the quantum effects, one should make sure that rapid classical instabilities are also removed. These may require further modification of the model. Detailed studies of this issue will be presented elsewhere.

Having the formalism of the previous subsection developed it is easy now to discuss the spectrum of massive theory. First we restore the gauge invariance of the massive Lagrangian (103) by using the variables (99), and then gauge fix it by (101). The resulting Lagrangian is

$$l = \frac{1}{2}{\rm Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda\partial^\mu\left(\frac{U^+D_\mu U}{ig}\right) + \frac{M^2}{2}\left(\frac{U^+D_\mu U}{ig}\right)^2 + \frac{1}{2\xi}(\partial_\mu B^\mu - \xi\lambda)^2\right) \quad (108)$$

The mixing term between the gauge field and Goldstones is canceled and we integrate out the λ field. The resulting Lagrangian reads:

$$l = \frac{1}{2}\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}(A) + \frac{1}{2}M^2 A_\mu^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{\xi}(\partial_\mu A^\mu)(\partial^\mu \Pi_\mu) + A_\mu J^\mu\right), \quad (109)$$

where, as before, we defined $\Pi_\mu = UA_\mu U^+ + \frac{i}{g}[\partial_\mu U]U^+ - A_\mu$.

A remarkable feature of this model is that the propagator takes the form:

$$\Delta^{ab}_{\mu\nu} = -\delta^{ab}\left(\eta_{\mu\nu} - \frac{(1+\xi)}{\Box - \xi M^2}\partial_\mu\partial_\nu\right)\frac{1}{\Box + M^2} - \delta^{ab}\frac{\xi\partial_\mu\partial_\nu}{(\Box - \xi M^2)\Box}, \quad (110)$$

which has a smooth UV behavior and non-singular massless limit.

References

[1] N. Nakanishi, Prog. Theor. Phys. **35** (1966) 1111.

[2] B. Lautrup, Mat. Fys. Madd. Dan. Vid. Selsk. **35** (1967) 1.

[3] K. Symanzik, *"Lectures on Lagrangian Quantum Field Theory"*, edited by H.-J. Thun, University of Islamabad, Pakistan, 1968.

[4] T. Kugo and I. Ojima, Phys. Lett. B **73**, 459 (1978). Prog. Theor. Phys. **60**, 1869 (1978). Prog. Theor. Phys. **61**, 294 (1979).
T. Kugo and I. Ojima, Nucl. Phys. B **144**, 234 (1978).

[5] C. Becchi, A. Rouet and R. Stora, Annals Phys. **98**, 287 (1976).
I.V. Tyutin, Lebedev Institute Preprint, Moscow, (1975).

[6] J. J. van der Bij, H. van Dam and Y. J. Ng, Physica **116A**, 307 (1982).
F. Wilczek and A. Zee, presented by A. Zee, in "High Energy Physics", eds. B. Kursunoglu et al. Plenum press, New York, (1985).
W. Buchmuller and N. Dragon, Phys. Lett. B **207** (1988) 292.

[7] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[8] G. Gabadadze, Y. Shang, in preparation.

[9] J. A. Frieman, C. T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. **75**, 2077 (1995) [arXiv:astro-ph/9505060].

[10] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, arXiv:hep-th/0209227.

[11] J. F. Donoghue, Talk given at Advanced School on Effective Theories, Almunecar, Spain, 25 Jun - 1 Jul 1995. arXiv:gr-qc/9512024.

[12] N. Goldberg, Phys. Rev. **111**, 315 (1958)

[13] R. Delbourgo and M. Ramon Medrano, Nucl. Phys. B **110**, 467 (1976).

[14] A. A. Slavnov, Theor. Math. Phys. **10** (1972) 99 [Teor. Mat. Fiz. **10** (1972) 153].
J. C. Taylor, Nucl. Phys. B **33** (1971) 436.

[15] D. M. Capper, G. Leibbrandt and M. Ramon Medrano, Phys. Rev. D **8** (1973) 4320.

[16] P. K. Townsend and P. van Nieuwenhuizen, Nucl. Phys. B **120**, 301 (1977).

[17] W. Buchmuller and N. Dragon, Phys. Lett. B **223**, 313 (1989).

[18] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173** (1939) 211.

[19] D. G. Boulware and S. Deser, Phys. Rev. D **6** (1972) 3368.

[20] G. Gabadadze and A. Gruzinov, arXiv:hep-th/0312074.

[21] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, arXiv:hep-th/0505147.

[22] C. Deffayet and J. W. Rombouts, arXiv:gr-qc/0505134.

[23] G. Gabadadze and M. Shifman, Phys. Rev. D **69**, 124032 (2004) [arXiv:hep-th/0312289].

[24] M. Porrati and J. W. Rombouts, Phys. Rev. D **69**, 122003 (2004) [arXiv:hep-th/0401211].

[25] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000) [arXiv:hep-th/0005016].

[26] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002) [arXiv:hep-th/0106001].

[27] A. Gruzinov, arXiv:astro-ph/0112246.

[28] A. Lue and G. Starkman, Phys. Rev. D **67**, 064002 (2003) [arXiv:astro-ph/0212083].

[29] G. Gabadadze and A. Iglesias, arXiv:hep-th/0407049.

[30] G. Dvali, arXiv:hep-th/0410286.